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2000 J. Phys. A: Math. Gen. 33 5457

(http://iopscience.iop.org/0305-4470/33/31/301)

J. Phys. A: Math. Gen. **33** (2000) 5457–5463. Printed in the UK PII: S0305-4470(00)12642-3

The simplification of the loop-counting method for the 2D Ising model

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Received 14 March 2000

Abstract. The proof of the loop-counting method for the 2D Ising model is simplified by analysing the root of the determinant of the transfer matrix. The one-to-one correspondence between the partition function and the root is obtained in a way which is more transparent than in other approaches. In the proposed method the vanishing of certain classes of loop diagrams takes place automatically.

1. Introduction

The well known combinatorial method of solving the 2D Ising model was introduced by Kac and Ward [1] and later improved by Vdovichenko [2]. Unfortunately, in the literature there exist only a few complete proofs of the method. The first proof was given by Sherman [3] and Burgoyne [4], while the simplest proof so far was given by Morita [5]. In all approaches one faces the problem of proving one-to-one correspondence between the partition function, which is represented by the sum of closed polygons, and the Taylor expansion of the root of the special determinant. Unfortunately, this correspondence is clearly seen only for very simple diagrams. The Taylor expansion contains very complicated diagrams and not all of them correspond to closed polygons. This causes the necessity of proving the cancellation of a broad class of diagrams. Our point is that in order to complete the proof one only has to prove the cancellation of a subset of these diagrams. The remaining diagrams cancel automatically due to the fact that the square of the considered expression is equal to the special determinant. Although this method does not allow one to find the solution for the 3D or 2D lattice in a nonzero external field, the proposed proof of cancellation of diagrams may be applied if the partition function of some model can be expressed by the root of a determinant.

2. Diagrammatic theorem

Every planar Ising lattice can be transformed into the rectangular lattice by attaching suitable coupling constants to the bonds connecting spins and—if necessary—adding auxiliary spins. Thus every partition function of the planar lattice can be written in the form

$$Z = \sum_{\{\sigma_{ij}\}} \prod_{ij} \exp(\beta J_{1ij}\sigma_{ij}\sigma_{i+1j} + \beta J_{2ij}\sigma_{ij}\sigma_{ij+1}) \qquad (1)$$

or equivalently

$$Z = \sum_{\{\sigma_{ij}\}} \prod_{ij} ((1 - K_{ij}^2)(1 - L_{ij}^2))^{-\frac{1}{2}} (1 + K_{ij}\sigma_{ij}\sigma_{i+1j})(1 + L_{ij}\sigma_{ij}\sigma_{ij+1}). \qquad (2)$$

J_{1ij}, J_{2ij} are the coupling constants in the horizontal and vertical directions, $K_{ij} = \tanh(\beta J_{1ij})$, $L_{ij} = \tanh(\beta J_{2ij})$ and $\sigma_{ij} = \pm 1$. The sum is taken over all spin configurations. In case of periodic boundary conditions indices i, j are cyclic. The quantity z, which will be essential for further analysis, is defined in the following way:

$$Z = z \prod_{ij} 2((1 - K_{ij}^2)(1 - L_{ij}^2))^{-\frac{1}{2}}. \qquad (3)$$

By splitting Z into a sum of products one is left with only those terms in which all powers of spin variables are even. Therefore z is the sum of all different closed diagrams, created by drawing polygons on the lattice in such a way that only an even number of bonds enters each lattice site. The weight of a given diagram is equal to the product of all coupling constants attached to the bonds of the diagram.

To calculate these diagrams we shall construct the special *transfer matrix* V using the formalism proposed by Vdovichenko [2]. Instead of considering the space of spin configurations one considers the space of *directed bonds*. Each bond can be denoted as $|kl\nu\rangle$ where k and l denote the starting point of the bond (lattice site) and ν the direction ($\nu = 1, \ldots, 4$). Numbers $\nu = 1, 2, 3, 4$ correspond to directions from [0, 0] to [1, 0], [0, 1], [−1, 0] and [0, −1], respectively. The maximum dimension of the space is four times the number of spins. The transfer matrix V, which acts in this space, must satisfy the following conditions.

- It transforms the given bond into one with the starting point located where the direction of the given bond points.
- It does not allow us to transform the given bond into one with opposite direction.
- The weight of a matrix element is equal to the coupling constant (K_{ij} or L_{ij}) attached to the given bond.
- For each matrix element one introduces the additional weight chosen in such a way that the product of matrix elements associated with a loop without intersections is equal to −1.

The schematic structure of V is shown in figure 1. The following form of the transfer matrix elements is possible:

$$\begin{aligned}
V|kl1\rangle &= K_{k,l}(|k+1, l, 1\rangle - |k+1, l, 2\rangle + |k+1, l, 4\rangle) \\
V|kl2\rangle &= L_{k,l}(-|k, l+1, 1\rangle + |k, l+1, 2\rangle + |k, l+1, 3\rangle) \\
V|kl3\rangle &= K_{k-1,l}(|k-1, l, 2\rangle + |k-1, l, 3\rangle + |k-1, l, 4\rangle) \\
V|kl4\rangle &= L_{k,l-1}(|k, l-1, 1\rangle + |k, l-1, 3\rangle + |k, l-1, 4\rangle).
\end{aligned} \qquad (4)$$

It is straightforward to check that all changes of the sign are caused by elements of the type $\langle i+1j2|V|ij1\rangle$ and $\langle ij+11|V|ij2\rangle$, whose number is always odd for a loop with no intersections. Thus the last condition in the list above is fulfilled.

Our main purpose is to prove the following theorem.

Diagrammatic theorem.

$$z = [\det(I - V)]^{\frac{1}{2}} \qquad (5)$$

where z and V are defined by (3) and (4) for the planar lattice.

Remark. The square root in equation (5) is single valued only for small values of K_{ij} and L_{ij}. However, this constraint does not affect the proof since we will treat this expression only

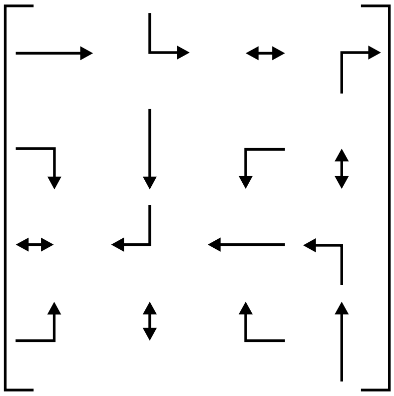

Figure 1. The structure of matrix V.

as a formal series. Nevertheless, in the explicit calculation of this expression we must treat it as an analytic extension of the function defined by the series at $K_{ij} = 0$, $L_{ij} = 0$. This is especially important in the case of periodic boundary conditions. (For a typical problem related to crossing the critical point see [6].)

3. Proof of the diagrammatic theorem

Every term in the series expansion of $[\det(I - V)]^{\frac{1}{2}}$ has the form

$$\omega \prod_{ij} K_{ij}^{\alpha(ij)} L_{ij}^{\beta(ij)} \tag{6}$$

where ω is independent of K_{ij}, L_{ij} but depends on the exponents $\alpha(ij), \beta(ij) \in \boldsymbol{N}$. Expression $\det(I - V)$ is also a sum of terms of this form. The following lemmas play a crucial role in our simplification of the loop-counting method. They will allow us to exclude from consideration all terms where $\exists i, j : \alpha(ij) > 2$ or $\beta(ij) > 2$.

Lemma 1. *In the series expansion of* $\det(I - V)$ *in variables* K_{ij} *and* L_{ij} *all nonzero terms of the form (6) satisfy*

$$\forall i, j : \alpha(ij) \qquad \beta(ij) \in \{0, 1, 2\}. \tag{7}$$

Proof. The determinant is a sum of such products of matrix elements that two elements associated with the same row or column cannot be present simultaneously in the same product. The only elements of V containing the same K_{ij} but referring to a different row and column have the form $\langle i+1\, j \mu | V | i j 1 \rangle$ and $\langle i j \nu | V | i+1\, j 3 \rangle$ (similarly for L_{ij}). □

Lemma 2. *If there exists any term of the form (6) in the series expansion of* $[\det(1 - V)]^{\frac{1}{2}}$ *such that*

$$\exists i, j : \alpha(ij) > 2 \qquad \text{or} \qquad \beta(ij) > 2 \tag{8}$$

then there exists an element in this series satisfying (7), in which

$$\exists i, j : \alpha(ij) = 2 \qquad \text{or} \qquad \beta(ij) = 2. \tag{9}$$

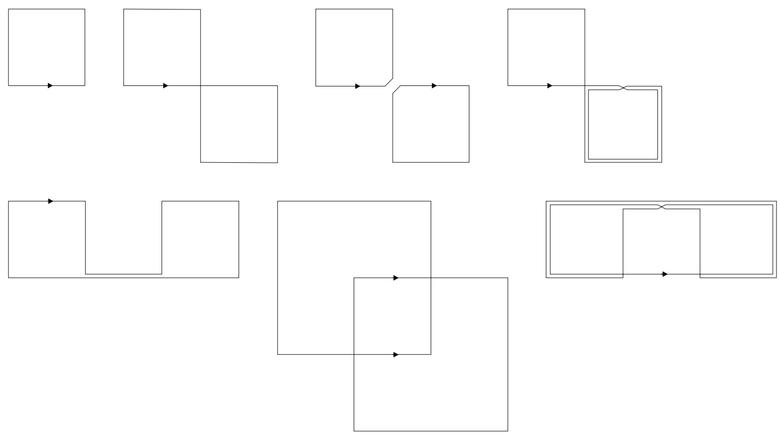

Figure 2. Examples of diagrams.

Proof. Let us take a term satisfying (8) such that

$$\sum_{ij}(\alpha(ij) + \beta(ij)) \text{ is the least for this group of terms.} \tag{10}$$

If there is more than one such term then one takes any of them. Suppose that there are no terms satisfying (7) and (9). After taking the square of the whole expression we obtain a sum of all products of two terms of the expansion of the square root. Multiplication of two terms, each satisfying $\forall i, j : \alpha(ij) < 2$ and $\beta(ij) < 2$, can never lead to a product satisfying (8). Moreover, the only way to obtain the term satisfying (8) and (10) (with double weight) is to multiply unity by this term. However, the previous lemma implies that no such terms exist in the expansion of the determinant. □

The two above lemmas make it clear that we need to consider only terms for which $\forall i, j : \alpha(ij), \beta(ij) \leqslant 2$ and to show that there are no terms satisfying (9).

Below the following expansion will be used:

$$[\det(I - V)]^{\frac{1}{2}} = \exp\left(-\tfrac{1}{2}\sum_{n=1}^{\infty} \operatorname{tr} V^n/n\right)$$
$$= \sum_{(s_j)\geqslant 0}^{\infty}\prod_{n=1}^{\infty}(-1)^{s_n}(\operatorname{tr} V^n)^{s_n}/[(2n)^{s_n}s_n!]. \tag{11}$$

In equation (11) tr V^n is a sum of products of matrix elements representing directed paths on the lattice. The paths begin and end as lines with the same starting point and direction (due to the trace operation): they form *connected diagrams*. Every connected diagram of this sum is a product of n suitable coupling constants and has an appropriate sign.

It follows from equation (4) that the sign of every connected diagram without intersections is equal to -1. It can be checked (see also [7]) that the sign of any connected diagram is equal to $(-1)^{m+1}$, where m is a number of intersections.

The rhs of equation (11) is a sum of products of connected diagrams, which will be from now on denoted as *diagrams*. Figure 2 presents various possible structures of diagrams. Connected parts intersect one another at an even number of points. Because there is a minus sign for every connected part of a diagram the resulting sign of a diagram in equation (11) is $(-1)^m$, where m is the number of all intersections.

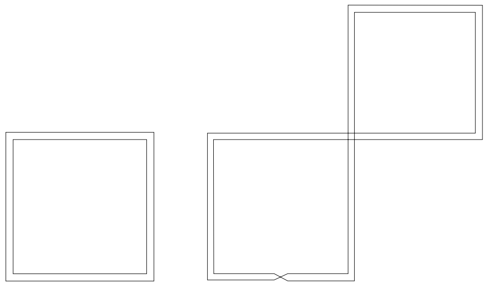

Figure 3. Examples of double loops (undirected).

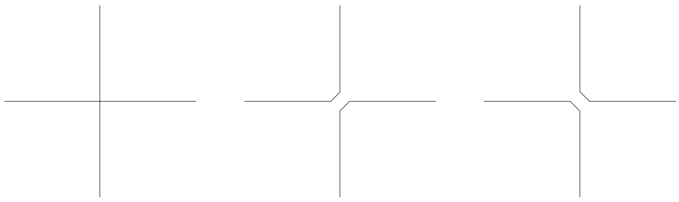

Figure 4. Three connections of bonds.

Now we consider only those diagrams contributing to equation (11) which satisfy $\forall i, j : \alpha(ij), \beta(ij) \leqslant 2$. The denominator 2^{s_n} is equal to the number of ways of attaching directions to diagrams. By considering only *undirected diagrams* one gets rid of the above denominator. An important example of undirected diagrams is a *double loop*, which is obtained by placing two identical *undirected connected diagrams* on each other or it becomes such after changing the connection of bonds at one point (in the middle of the bond, not at a lattice site, figure 3).

Remark. All connections are made in such a way that when a certain bond is passed several times then the ends of the bond are copied a corresponding number of times and lines can be drawn from the first to the second end. Therefore there are only two possibilities of connecting the ends in a twice-passed bond: parallel or crossing.

The denominator $s_n!$ takes into account the number of permutations. n takes into account the fact that connected diagrams of tr V^n have as many copies as there are possible starting bonds. Double loops are exceptions. In the first case they leave denominator 2! untouched, in the second starting bonds are counted twice. Therefore the weight of diagram of equation (11) satisfying (7) is $\frac{1}{2^d}$, where d is the number of double loops in a given diagram.

For a given diagram with a bond passed twice there always exists another diagram with the number of intersections differing by one. Thus these two diagrams cancel each other. For this reason according to lemma 2 only diagrams with bonds passed once survive. There are now three possible connections of bonds in the lattice site (figure 4). They correspond to three constructions of a diagram with the same weight but different signs $(-, +, +)$. Only one diagram without an intersection at the lattice site survives. When this reasoning is repeated for every site where four bonds of a diagram meet, one is left with only single diagrams without intersections. They are equivalent to products of closed polygons drawn on the lattice, and they give only products of coupling constants attached to bonds of polygons. Their sum is equal to z as shown in the previous section.

4. Periodic boundary conditions

The above theorem works only for planar lattices, but there exists a method of solving the problem with periodic boundary conditions introduced by Potts and Ward [6]. One introduces three matrices V_i, $i = 1, 2, 3$:

$$\begin{aligned}
V_1|kl1\rangle &= e^{i\pi/M} V|kl1\rangle & V_1|kl2\rangle &= e^{i\pi/N} V|kl2\rangle \\
V_1|kl3\rangle &= e^{-i\pi/M} V|kl3\rangle & V_1|kl4\rangle &= e^{-i\pi/N} V|kl4\rangle \\
V_2|kl1\rangle &= e^{i\pi/M} V|kl1\rangle & V_2|kl2\rangle &= V|kl2\rangle \\
V_2|kl3\rangle &= e^{-i\pi/M} V|kl3\rangle & V_2|kl4\rangle &= V|kl4\rangle \\
V_3|kl1\rangle &= V|kl1\rangle & V_3|kl2\rangle &= e^{i\pi/N} V|kl2\rangle \\
V_3|kl3\rangle &= V|kl3\rangle & V_3|kl4\rangle &= e^{-i\pi/N} V|kl4\rangle
\end{aligned} \tag{12}$$

where M and N are the sizes of the lattice in directions $[x, 0]$ and $[0, y]$.

The sign of the simple loop (*cycle*) in direction $[x, 0]$ or $[0, y]$ which can occur in tr V^n is $+1$ since there are no turns. One introduces numbers a and b which count how many times the diagram winds the torus in direction $[x, 0]$ and $[0, y]$. The parity of a and b is independent of any deformation or the method of connections at the point where four bonds meet. One can now take any diagram in equation (11) and deform it to obtain the product of simple loops. Each simple loop changes the sign. However, the ab intersections must be taken into account and therefore the sign of a diagram in equation (11) is $(-1)^{m+ab+a+b}$, where m is the number of intersections. Matrices V_i, $i = 1, 2, 3$ change this sign by $(-1)^{a+b,a,b}$ respectively, so that four groups of diagrams (for different parity of a and b) have different signs. Let z_{kl} be the sum of diagrams with the parity of the number of cycles in direction $[x, 0]$ and $[0, y]$ equal to the parity of k and l where $k, l \in \{0, 1\}$. One obtains the equations for z_{kl}:

$$\begin{aligned}
z_{00} - z_{10} - z_{01} - z_{11} &= [\det(I - V)]^{\frac{1}{2}} \\
z_{00} + z_{10} + z_{01} - z_{11} &= [\det(I - V_1)]^{\frac{1}{2}} \\
z_{00} + z_{10} - z_{01} + z_{11} &= [\det(I - V_2)]^{\frac{1}{2}} \\
z_{00} - z_{10} + z_{01} + z_{11} &= [\det(I - V_3)]^{\frac{1}{2}} \\
z_{00} + z_{10} + z_{01} + z_{11} &= z.
\end{aligned} \tag{13}$$

Solving these equations we obtain the exact expression for z:

$$z = \frac{1}{2}\left(\sum_{i=1}^{3}[\det(I - V_i)]^{\frac{1}{2}} - [\det(I - V)]^{\frac{1}{2}}\right). \tag{14}$$

5. Conclusion

When proving the diagrammatic theorem we took the advantage of using the root of a determinant. This allowed us, in contrast to the previous papers, to avoid the analysis of very complicated diagrams. In our approach they cancel automatically. Then it is only necessary to analyse a small group of diagrams (with bonds appearing finite number of times in a diagram). This makes this method applicable to other cases in which the partition function of certain model is represented as the root of low degree of a determinant. Then one may check the one-to-one correspondence between the diagrams in the partition function and the terms of the Taylor expansion in a very effective way. Moreover, this shows clearly that the combinatorial method of solving the planar Ising-type models is the easiest one.

Acknowledgments

I am grateful to Professor M Napiórkowski and J Wojtkiewicz for a careful reading of the manuscript and G Łach for encouragement to do this work.

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